Correction R-commutes with $P_n^{\triangle} = P_n^{(1)} + P_n^{(2)} + \cdots$

- This is trivial.

$$X = \coprod_d \widetilde{\mathfrak{M}}_r^d \qquad
\begin{tikzcd}
 & & H^*_{\mathbb{T}}(X) \arrow[d, phantom, "\shortparallel"] & & \\
H^*_{\mathbb{T}}(X^T) \arrow[r, "\mathrm{Stab}"] \arrow[loop below, "P_n^{\triangle}"] & H^{\mathbb{T}}_*(\mathcal{A}_X) \arrow[r] & H^{\mathbb{T}}_*(X) \arrow[loop below, "P_n^{\triangle}"] & H^{T}_*(\mathcal{A}_X^{op}) \arrow[l] & H^*_{\mathbb{T}}(X^T) \arrow[l, "\mathrm{Stab}'"'] \arrow[loop below, "P_n^{\triangle}"]
\end{tikzcd}$$

Defined via Stab

original

In the same way, $c_k(\mathcal{V})$: Chern class of $\mathcal{V}$ tautological bdle

$\left(\mathbb{E}|_{(E,\varphi)} = H^1(E(-l_\infty)) \right.$ or $\mu^{-1}(0) \times_{GL(V)} V$ in quiver var.

commutes with R

- **Addendum**

$Z_\sigma = \mathcal{A}_X \times_{X_0^T} X^T$ $\qquad$ $Z_{\sigma^-} = X^T \times_{X_0^T} \mathcal{R}_X$ $\overset{\shortparallel}{}$ $\mathcal{A}_X$ for $-\mathfrak{C}$ or $\lim_{t\to\infty}$

$(Z = X \times_{X_0} X)$

$$H^{\mathbb{T}}_*(Z_{\sigma^-}) \otimes H^{\mathbb{T}}_*(Z) \otimes H^{\mathbb{T}}_*(Z_\sigma) \longrightarrow H^{\mathbb{T}}_*(Z^T)$$

§ Key points of the proof of Th [MO]

$R =$ reflection operator

1°. How to compute R ?

2°. Why Virasoro alg. appears ?

2°. $\leftarrow$ Th [Lehn '99]

For Hilb^d - case

$$[c_1(\mathcal{V}), P_n^{\Delta}] = \frac{n}{2} \sum \underbrace{:P_m P_{n-m}:}_{\text{Virasoro (ordinary)}} - \frac{n(|n|-1)}{2} (\varepsilon_1 + \varepsilon_2) P_n$$

[MO] : higher rank generalization

(The strange $|n|$ term does not affect the argument)

$\longrightarrow$ come back later

1°

- $[R, P_n^{\Delta}] = 0 \Rightarrow$ Enough to compute

$$\big(\underset{\mathrm{Stab}_{-\mathfrak{C}}^{-1}\circ \mathrm{Stab}_{\mathfrak{C}}}{R}(\underbrace{\mathrm{vac}\otimes \mathrm{Fock}}_{\subset\ \mathrm{Hilb}^0\times \mathrm{Hilb}^d}),\ \mathrm{vac}\otimes\mathrm{Fock}\big)$$

- $\mathrm{Stab}_{-\mathfrak{C}}^{-1} = (-1)^{\frac{1}{2}\mathrm{codim}\, X^T}\, \mathcal{L}^t$ transpose

∵ Recall $\mathcal{L} \subset \mathcal{A}_X \times_{X_0^T} X^T \subset X \times X^T$

$$\begin{tikzcd} & \mathcal{A}_X \times_{X_0^T} X^T \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathcal{A}_X & & X^T \end{tikzcd}$$

$p_{1*}(p_2^*(\cdot) \cap \mathcal{L})$

$\mathcal{L}^t = p_{2*}(p_1^*(\cdot) \cap \mathcal{L})$ p_2: not proper

or use app. coh

(equivariant int, OK.)

$$\underset{X^T\times \mathcal{A}_X}{\mathrm{Stab}_{\mathfrak{C}}^t} \circ \underset{\substack{\mathcal{R}_X \times X^T \\ \| \\ \mathcal{A}_X^{op}}}{\mathrm{Stab}_{-\mathfrak{C}}} = p_{13*}(p_{12}^*\mathcal{L}^t \cap p_{23}^*\mathcal{L}) = \Delta_{X^T}$$

$$\begin{tikzcd} & X^T\times X \times X^T \arrow[dl, "p_{12}"'] \arrow[dr, "p_{23}"] & \\ X^T\times X & & X\times X^T \end{tikzcd}$$

$\therefore\ (R(\mathrm{vac}\otimes F), \mathrm{vac}\otimes F) = (\mathrm{Stab}_{\mathfrak{C}}(\mathrm{vac}\otimes F), \mathrm{Stab}_{\mathfrak{C}}(\mathrm{vac}\otimes F))$

$$R = \mathrm{Stab}_{-\mathfrak{C}}^{-1}\circ \mathrm{Stab}_{\mathfrak{C}}, \qquad \mathrm{Stab}_{-\mathfrak{C}}^{-1} = \mathrm{Stab}_{\mathfrak{C}}^t$$

$$(\widetilde{\mathcal{U}}_2^d)^{\mathbb{C}^*} = \coprod_{d=d_1+d_2} \mathrm{Hilb}^{d_1} \times \mathrm{Hilb}^{d_2}$$

Lemma $\mathrm{Hilb}^0 \times \mathrm{Hilb}^d$ is the minimum element.

(like $\times \,\vdots\, \overset{\bullet}{\infty}$)

$\Rightarrow$ $\mathrm{Stab}_{\mathcal{C}}(\mathrm{vac} \otimes F) = \mathrm{Leaf}$ for $\mathrm{Hilb}^0 \times \mathrm{Hilb}^d$

$$T_{(E,\varphi)} \widetilde{\mathcal{U}}^d_2 = \mathrm{Ext}^1(E, E(-l_\infty)) \qquad E = I_1 \oplus I_2$$

$$= \bigoplus_{\alpha,\beta=1}^{2} \mathrm{Ext}^1(I_\alpha, I_\beta(-l_\infty)) \oplus$$

$$(\alpha,\beta) = (1,1) \quad T_{I_1}\mathrm{Hilb}^{d_1}$$
$$(2,2) \quad T_{I_2}\mathrm{Hilb}^{d_2}$$

normal bdle $=$ mixed term

$$\Rightarrow \mathrm{Leaf} = \mathrm{Ext}^1(I_2, I_1(-l_\infty))$$
$$(\mathrm{Leaf}^- = \mathrm{Ext}^1(I_1, I_2(-l_\infty)))$$

$\therefore$ $(\mathrm{Stab}_{\mathcal{C}}(\mathrm{vac} \otimes F), \mathrm{Stab}_{\mathcal{C}}(\mathrm{vac} \otimes F))$

$$= \int_{\mathrm{Hilb}^0 \times \mathrm{Hilb}^d} \frac{e(\mathrm{Leaf}^-)^2}{e(N)} = \int_{\mathrm{Hilb}^0 \times \mathrm{Hilb}^d} \frac{e(\mathrm{Leaf}^-)\cancel{e(\mathrm{Leaf}^-)}}{e(\mathrm{Leaf}^+)\cancel{e(\mathrm{Leaf})}}$$

$\mathrm{Leaf}^- = (\mathrm{Leaf})^\vee$ by symplectic form

invariant under T

but not under $\mathbb{T}$

$$\mathrm{Leaf}^- = \mathrm{Leaf} \otimes \underset{t_1 t_2}{\overset{\shortparallel}{e^{\varepsilon_1+\varepsilon_2}}} \qquad t^* \omega = t_1 t_2 \, \omega$$

$$= \int_{\mathrm{Hilb}^d} \frac{e(\mathcal{V} \otimes e^{\varepsilon_1+\varepsilon_2})}{e(\mathcal{V})} \longrightarrow \text{Chern classes of } \mathcal{V}.$$

§ Intersection cohomology of Uhlenbeck spaces

[Chriss-Ginzburg. Chap 8]

$D^b(X) = D^b_{cons}(X)$: derived category of constructible sheaves on X

$f: X \to Y$ $\quad f_!, f_*, f^*, f^!, \otimes^L$, $R\mathcal{H}om$, $\mathbb{D}_X$: Verdier duality

$\pi : X \to X_0$ proper (smooth X)

decomposition thm [BBDG]

$$\pi_* \mathbb{C}_X \cong \bigoplus IC(X_\alpha, L_\alpha)[d_\alpha]^{\oplus m_\alpha} \quad \text{(semisimple cpx)}$$

$$\pi_* \mathbb{C}_X = \pi_! \mathbb{C}_X$$

<u>Def</u>. (Borho-MacPherson)

$\pi : X \to X_0$ semismall $\Leftrightarrow$ $\exists\, X_0 = \bigsqcup X_\alpha$ stratification

s.t. $f|_{f^{-1}(X_\alpha)}$ top. fibration

& $\dim f^{-1}(x_\alpha) \leqq \frac{1}{2} \operatorname{codim} X_\alpha$

$$\Longrightarrow \quad \pi_* \underline{\mathbb{C}_X[\dim X]} \cong \bigoplus_\alpha IC(X_\alpha, \chi) \otimes H_{top}(\pi^{-1}(x))_\chi$$

$\mathcal{C}_X$ (= $\mathbb{C}_X[\dim X]$), χ: local system, $x \in X_\alpha$

without <u>shift</u>

loc. system over X_α: $\underline{H_{top}(\pi^{-1}(x))} = \bigoplus H_{top}(\pi^{-1}(x))_\chi \otimes \chi$

<u>Fact</u> [Kaledin] symplectic resolution is semismall

π^T, π : semismall $\quad \therefore$ $\pi^T_* \mathbb{C}_{X^T}$, $\pi_* \mathbb{C}_X$: s.s. Perv. sheaf (without shift)

Rem. $\pi : \widetilde{\mathcal{U}}_r^d \longrightarrow \mathcal{U}_G^d$ $\qquad (E,\varphi) \mapsto (E^{\vee\vee},\varphi), \mathrm{Supp}\, E^{\vee\vee}/E$

$$2dr \qquad \mathcal{U}_G^d = \coprod_{|\lambda|+d'=d} \mathrm{Bun}_G^{d'} \times S_\lambda \mathbb{A}^2$$

$$\uparrow 2\ell(\lambda)+2d'r$$

$$\mathrm{codim} = 2(d-d')r - 2\ell(\lambda) = 2|\lambda| r - 2\ell(\lambda)$$

$\mathfrak{M}_0(V,W)$: semisimple rep

simple $\iff$ stable & costable ($\iff$ transpose is stable)
or $W=0$ • $[B_1,B_2]=0$
$\Rightarrow \dim V = 1$ $\quad B_1, B_2$: scalar

Fact $\dim \pi^{-1}(x) = \frac{1}{2}\mathrm{codim}\, X_\alpha \qquad x \in X_\alpha$
$\pi^{-1}(x)$: irreducible

This is used in Baranovsky's result.

$[\pi^{-1}(d\cdot 0)]$ $\qquad d\cdot 0 \in S^d_{(d)}\mathbb{A}^2 \leftarrow$ most singular stratum

$\mathrm{codim} = 2dr-2$

$$H^{\mathrm{ord},T}_{2(dr-1)}(\widetilde{\mathcal{U}}_r^d) \overset{PD}{\cong} H^*_{T,c}(\widetilde{\mathcal{U}}_r^d)$$

$$P^{\Delta}_{-d}([0]) \cdot \underline{|\mathrm{vac}\rangle} \qquad 1 \in H^0_T(\widetilde{\mathcal{U}}_r^0 = pt)$$

$$\therefore \quad \pi_* \mathbb{C}_{\widetilde{\mathcal{U}}_r^d} = \bigoplus_{|\lambda|+d'=d} IC(\mathrm{Bun}_G^{d'} \times S_\lambda \mathbb{A}^2) \otimes H_{[0]}(\pi^{-1}(d\cdot 0))$$

$$\Rightarrow \quad H_T^*(\widetilde{\mathcal{U}}_r^d) = IH_T^*(\mathcal{U}_G^d) \oplus \text{smaller}$$

$$\uparrow d'=d$$

moreover smaller pieces can be understood recursively.

$$\xi : \mathcal{U}_G^{d'} \times \underbrace{\mathbb{A}^2 \times \cdots \times \mathbb{A}^2}_{\ell(\lambda)} / \mathrm{Stab}(\lambda) \longrightarrow \overline{\mathrm{Bun}_G^{d'} \times S_\lambda \mathbb{A}^2} \subset \mathcal{U}_G^d$$

extend to a finite morphism

$$\cup \qquad\qquad\qquad \cup$$

$$\mathrm{Bun}_G^{d'} \times S_\lambda \mathbb{A}^2 \xrightarrow{\cong}$$

$$IC(\mathrm{Bun}_G^{d'} \times S_\lambda \mathbb{A}^2) = \xi_*\left(IC(\mathrm{Bun}_G^{d'}) \boxtimes \mathbb{C}_{\mathbb{A}^2 \times \cdots \times \mathbb{A}^2 / \mathrm{Stab}(\lambda)}\right)$$

$$\Longrightarrow \bigoplus_d H^*_{\mathbb{T}}(\tilde{\mathcal{U}}_r^d) \cong \bigoplus_d IH^*_{\mathbb{T}}(\mathcal{U}_G^d) \otimes \left(\begin{array}{c}\text{Fock space} \\ \text{for } P_n^{\Delta}(1)\end{array}\right)$$

On $IH^*_{\mathbb{T}}(\mathcal{U}_G^d)$, the mysterious part disappear:

$$[c_1(\mathcal{V}), P_n^{\Delta}]\Big|_{IH^*_{\mathbb{T}}(\mathcal{U}^d_{SL_2})} = n L_n \quad (FF)$$

$r = 2 \Rightarrow$ We get enough big algebra!

$$\bigoplus_d H^*_{\mathbb{T},c}(\tilde{\mathcal{U}}_2^d) \cong \bigoplus_d IH^*_{\mathbb{T},c}(\mathcal{U}_G^d) \otimes \left(\begin{array}{c}\text{Fock space} \\ \text{for } P_n^{\Delta}[0]\end{array}\right)$$

Virasoro

$$\uparrow \text{Feigin-Fuchs}$$

$\mathrm{Heis}(P_n^-)$

$\hat{\mathfrak{gl}}_1 = \mathrm{Heis}.$

$$\Big\downarrow \mathrm{Stab}_{\mathfrak{C}}^{\varepsilon}$$

$$\bigoplus H^*_{\mathbb{T}}((\tilde{\mathcal{U}}_2^d)^T) \cong (\text{Fock for } P_n^-) \otimes (\text{Fock for } P_n^+)$$

- Sheaf theoretical analysis of convolution algebras

$$H_*(X \times_{X_0} X) \cong \mathrm{Ext}^*(\pi_* \mathbb{C}_X, \pi_* \mathbb{C}_X)$$

$$\pi_*\mathbb{C}_X \cong \bigoplus IC(X_\alpha, L_\alpha)[d_\alpha]^{\oplus m_\alpha} \quad \text{s.s. decomposition thm}$$

(sketch of the proof

$$\begin{tikzcd} X \times_{X_0} X \arrow[r, "i"] \arrow[d, "\pi\times\pi"'] & X \times X \arrow[d, "\pi\times\pi"] \\ \Delta X_0 \arrow[r, "\Delta"] & X_0 \times X_0 \end{tikzcd}$$

$$\begin{aligned} H_*(X \times_{X_0} X) &= H^*(X \times_{X_0} X, i^! \mathbb{C}_{X\times X}) \\ &= H^*(\Delta X_0, \underbrace{(\pi\times\pi)_* i^!}_{\Delta^!(\pi\times\pi)_*} \underbrace{\mathbb{C}_X \boxtimes \mathbb{C}_X}_{\mathbb{C}_X^\vee[\ ]}) \\ &= H^*(\Delta X_0, \mathcal{H}om(\pi_*\mathbb{C}_X, \pi_*\mathbb{C}_X)) \\ &= \mathrm{Ext}^*(\pi_*\mathbb{C}_X, \pi_*\mathbb{C}_X) \end{aligned}$$

- If π : semismall

$$\Longrightarrow H_{[0]}(X \times_{X_0} X) \cong \bigoplus \underbrace{\mathrm{End}(H_{top}(\pi^{-1}(x))_\chi)}_{\text{s.s. alg.}}$$

Our $Z_\mathfrak{g} = \mathcal{A}_X \times_{X_0^T} X^T$ is a variant of $X \times_{X_0} X$

$$\begin{tikzcd} X^T \arrow[d, "\pi^T"'] & & \mathcal{A}_X = \pi^{-1}(\mathcal{A}_{X_0}) \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ X_0^T & & \mathcal{A}_{X_0} \arrow[ll, "p"'] \arrow[r, "j"] & X_0 \end{tikzcd}$$

Prop [1207.0529]

$$H_*(Z_\mathfrak{g}) \cong \mathrm{Ext}^*(\pi^T_* \mathbb{C}_{X^T}, p_* j^! \pi_* \mathbb{C}_X)$$

proof) exercise

Want $H_{[0]}(Z_\mathfrak{g}) \cong \mathrm{Hom}(\pi^T_* \mathbb{C}_{X^T}, p_* j^! \pi_* \mathbb{C}_X)$

$\Longrightarrow \mathcal{L} : \pi^T_* \mathbb{C}_{X^T} \longrightarrow p_* j^! \pi_* \mathbb{C}'_X \quad (\cong)$

or $p_! j^* \pi_! \mathbb{C}_X \to \pi_! \mathbb{C}_{X^T}$

$\rightsquigarrow \mathrm{End}(\pi_! \mathbb{C}_X) \to \mathrm{End}(\pi_! \mathbb{C}_{X^T})$

$\therefore$ coproduct

But why $p_* j^!$(s.s. Perv. sheaf) still perverse?

2 parts
- direct sum of <u>shifts of</u> simple perverse sheaves
  $\leftarrow$ decomposition thm $\Leftarrow$ theory of wts
- shift does not appear
  $\leftarrow$ semismallness $\Leftarrow$ dimension estimates

Braden

Mirkovic-Vilonen

(or Varagnolo-Vasserot for symplectic resolution)

Exercise Use $\mathfrak{X}$ (deformation) to give another proof.

§ hyperbolic restriction

$X \curvearrowleft T$ Choose $\mathcal{C}$

$$X^T \underset{i}{\overset{p}{\leftrightarrows}} \mathcal{A}_X \xrightarrow{j} X$$

disjoint union according to conn. comp's of X^T

(We apply to X : affine)

Define $\Phi = p_* j^! : D^b_{const}(X) \longrightarrow D^b_{const}(Y)$

<u>Th</u> [Braden]

$p_* j^! \cong p_! j_-^*$ on weakly T-equivariant objects

($\mu : T \times X \to X$ action
$\mu^* \mathcal{F} \overset{\exists}{\cong} L \boxtimes \mathcal{F}$ loc. const. L)

where

$$X^T \underset{i_-}{\overset{p_-}{\leftrightarrows}} \mathcal{R}_X \xrightarrow{j_-} X$$

$\mathcal{R}_X = \mathcal{A}_X^{op}$ opposite chamber

<u>Cor</u>. Φ preserves the purity

So $IC(X)$ is sent to a direct sum of shifts of IC sheaves on X^T

$\because$ $p_*, j^!$ increase weights
$p_!, j^*$ decrease " //

Th [MV]

Assume $X_0^T = \{x\}$ and X has a stratification $X = \coprod X_\alpha$

s.t. $IC(X, \mathcal{L})|_{X_\alpha}$: loc. constant

$\dim \mathcal{A}_x \cap X_\alpha \leqq \frac{1}{2} \dim X_\alpha$

$\dim \mathcal{R}_x \cap X_\alpha \leqq \frac{1}{2}$ "

$$\begin{tikzcd} \mathcal{A}_x \arrow[r, "j"] \arrow[d, "p"] & X \\ \{x\} = X^T & \end{tikzcd}$$

$\Longrightarrow$ $p_* j^! (IC(X, L))$ is perverse

(proof)

$H^*(p_! j^* i^*_{X_\alpha} IC(X, L))$

$$\begin{tikzcd} \mathcal{A}_x \cap X_\alpha \arrow[r, "j"] \arrow[d, "p"] & X_\alpha \arrow[r, "i_x"] & X \\ \{x\} & & \end{tikzcd}$$

$= H_c^*(\mathcal{A}_x \cap X_\alpha, j^* i^*_{X_\alpha} IC(X, L))$

$x \in X_\alpha^T$ $\quad H_c^k(\mathcal{A}_x \cap X_\alpha) = 0$ if $k > 2\dim \mathcal{A}_x \cap X_\alpha$

$i^*_{X_\alpha} IC(X, L)$ has degree $\leqq -\dim X_\alpha$

$H_c^k(p^{-1}(x), (j \circ \tilde{\imath})^* i^*_{X_\alpha} IC(X, \mathcal{L})) \neq 0 \Rightarrow k \leqq 2\dim \mathcal{A}_x \cap X_\alpha - \dim X_\alpha$

$\leqq 0$

Assump.

dual $\Rightarrow k \geqq 0$ //

0 affine Grassmann

G : reductive group $/\mathbb{C}$ $\supset T$: max. torus

$\mathcal{O} = \mathbb{C}[[z]] \subset K = \mathbb{C}((z))$

$G(\mathcal{O}) \subset G(K)$

$\mathrm{Gr}_G := G(K)/G(\mathcal{O})$: affine Grassmann

$\lambda \in \mathrm{Hom}(\mathbb{C}^*, T)$ (coweight) $\rightsquigarrow$ point in Gr_G

$$\mathrm{Gr}_G = \coprod_{\lambda:\text{ dominant coweight}} \mathrm{Gr}_G^\lambda = \bigcup G_\mathcal{O}\cdot\lambda$$

$\mathrm{Perv}_{G_\mathcal{O}}(\mathrm{Gr}_G)$: category of $G_\mathcal{O}$-equivariant perverse sheaves on Gr_G

semisimple, $\{$simples$\} = IC(\overline{\mathrm{Gr}_G^\lambda})$

Th (Lusztig, Drinfeld, Ginzburg, Mirkovic–Vilonen)

$$\mathrm{Perv}_{G_\mathcal{O}}(\mathrm{Gr}_G) \cong \mathrm{Rep}(G^L) \qquad G^L : \text{Langlands dual}$$

$IC(\overline{\mathrm{Gr}_G^\lambda}) \leftrightarrow V(\lambda)$: irr. rep with h.w. $= \lambda$

convolution $\longleftrightarrow$ $\otimes$

[MV]: Weight space decomp. $V(\lambda) = \bigoplus_\mu V_\mu(\lambda)$ in the Gr_G-side.

Note $T \curvearrowright \mathrm{Gr}_G$ fixed pts $= \mathrm{Gr}_T = \mathrm{Hom}(\mathbb{C}^*, T)$ discrete $= \mathrm{Hom}(T^L, \mathbb{C}^*)$

Fix $\rho : \mathbb{C}^* \to T$ generic coweight (positive chamber) and define $\infty/2$-orbits

$$S_\mu := \{x \in \mathrm{Gr}_G \mid \lim_{t\to 0} \rho(t)x = \mu\}$$

$$T_\mu := \{x \in \mathrm{Gr}_G \mid \lim_{t\to \infty} \rho(t)x = \mu\}$$

$$p_* j^! : \mathrm{Perv}_{G_{\mathcal{O}}}(Gr_G) \longrightarrow D^b_{\mathrm{cons}}(Gr_T) = \bigoplus_{\mu \in \mathrm{Hom}(\mathbb{C}^*, T)} \underbrace{D^b(\mathrm{Vect})}_{\mathbb{Z}\text{-graded vect. sp.}}$$

$$p_* j^! = \bigoplus \Phi_\mu$$

<u>Prop</u> $\Phi_\mu(\mathrm{Perv}_{G_{\mathcal{O}}}(Gr_G))$ is concentrated in degree $2\langle \rho, \mu \rangle$

<u>key</u> of the proof: dimension estimate

$$\dim S_\mu \cap \overline{Gr^\lambda} = \langle \rho, \mu + \lambda \rangle$$
$$\dim T_\mu \cap \overline{Gr^\lambda} = -\langle \rho, \lambda + \mu \rangle$$

MV cycle : Irreducible components of $S_\mu \cap Gr^\lambda$
$\longrightarrow$ base of $V_\mu(\lambda)$

§ Uhlenbeck space

G: semisimple, simply-conn alg. grp $/\mathbb{C}$

$\mathcal{U}_G^d$: Uhlenbeck space

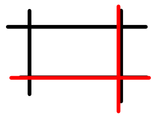
$$= \coprod \mathrm{Bun}_G^{d'} \times S_\lambda \mathbb{A}^2 \qquad \lambda \vdash (d-d')$$

$\mathbb{G} = G \times \mathbb{C}^* \times \mathbb{C}^*$

$\cup$

$\mathbb{T} = T \times$ "

Ex. $d=1$ $\mathcal{U}_G^1 = \mathbb{A}^2 \times \overline{\text{minimal nilpotent orbit of } \mathfrak{g}}$

- $\mathrm{Bun}_G^d = \{(E,\varphi) \mid \mathbb{P}^1 \times \mathbb{P}^1$ trivialised at  $\}$

$$\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{a} \mathbb{P}^1$$

<u>Alternative description</u> : $\mathrm{Bun}_G^d \cong \mathrm{Map}_d(\mathbb{P}^1, \mathrm{Gr}_G)$

affine Grassmann

space of based maps of degree d

- factorization

fix a projection $\mathbb{A}^2 \to \mathbb{A}^1$

$$\pi_a : \mathcal{U}_G^d \to S^d \mathbb{A}^1$$

$$\| $$

$$\coprod_{d',\lambda} \mathrm{Bun}_G^{d'} \times \underline{S_\lambda \mathbb{A}^2} \quad a$$

$a^{-1}(x) \cong \mathbb{P}^1$ $\qquad E|_{a^{-1}(x)}$: trivial when $x=\infty$

open condition

$\Longrightarrow$ $E|_{a^{-1}(x)}$: nontrivial at $x = x_1, \cdots, x_\ell$

If we count with multiplicities,

$\Longrightarrow$ # of pts $= d$

- fiber $\qquad \mathrm{Spec}(a_1 B_1 + a_2 B_2)$

$$a = [a_1 : a_2]$$

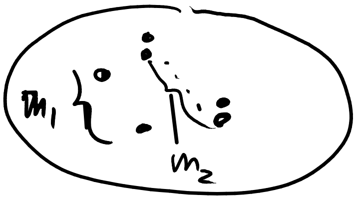
Factorization property :

$\pi_a(E,\varphi)$: disjoint $\Longrightarrow$ $\underset{\text{locally}}{\mathcal{U}_G^d \sim \mathcal{U}_G^{d_1}\times\cdots\times\mathcal{U}_G^{d_\ell}}$

§ class of perverse sheaves

$$\mathcal{U}_G^d = \coprod \underbrace{\mathrm{Bun}_G^{d'} \times S_\lambda \mathbb{A}^2}_{= \mathrm{Bun}^{d'}_{G,\lambda}} \qquad \lambda = (1^{m_1}2^{m_2}3^{m_3}\cdots)$$

$$\mathrm{Stab}\,\lambda = S_{m_1}\times S_{m_2}\times\cdots$$



χ : irrep. of $S_{m_1}\times S_{m_2}\times\cdots$

$\longrightarrow$ local system on $S_\lambda \mathbb{A}^2$

$\mathrm{Perv}(\mathcal{U}_G^d)$ = additive subcategory of perverse sheaves which are direct sums of

$$IC(\mathrm{Bun}^d_{G,\lambda},\chi)$$

reason

$G = SL_r$

$$\begin{tikzcd}
\coprod \mathrm{Hilb}^{d_1}\times\cdots\times\mathrm{Hilb}^{d_r} \arrow[d, "\pi\times\cdots\times\pi"'] & = (\widetilde{\mathcal{U}}_r^d)^T \arrow[d, "\pi^T"] & & \widetilde{\mathcal{U}}_r^d \arrow[d] \\
S^{d_1}\mathbb{A}^2\times\cdots\times S^{d_r}\mathbb{A}^2 \arrow[r, "\text{“+”}"'] & S^d\mathbb{A}^2 = \mathcal{U}_T^d & \mathcal{U}_B^d \arrow[l, "p"'] \arrow[r, "j"'] & \mathcal{U}_{SL_r}^d
\end{tikzcd}$$

$$p_* j^! \pi_* C_{\widetilde{\mathcal{U}}_r^d} \underset{\text{nontrivial}}{\cong} \pi^T_* C_{(\widetilde{\mathcal{U}}_r^d)^T}$$

$$\cong \text{“+”}_* \underbrace{(\pi\times\cdots\times\pi)_* C_{(\widetilde{\mathcal{U}}_r^d)^T}}_{=\left[\bigoplus C_{\overline{S_{\lambda_1}\mathbb{A}^2}}\boxtimes\cdots\boxtimes C_{\overline{S_{\lambda_r}\mathbb{A}^2}}\right]}$$

yields nontrivial local systems

Choose $\rho : \mathbb{C}^* \to T$ 1PS

$$L \twoheadleftarrow P \to G$$

$L := G^{\rho(\mathbb{C}^*)}$ Levi

$P := \{ g \in G \mid \lim_{t \to 0} \rho(t) g \rho(t)^{-1} \text{ exists} \}$ parabolic

e.g. $\rho(t) = \begin{bmatrix} \boxed{t^{m_1}} & & & \\ & t^{m_2} & & \\ & & \ddots & \\ & & & t^{m_k} \end{bmatrix}$ $m_1 > m_2 > \cdots > m_k$

ρ defines a $\mathbb{C}^*$-action on $\mathcal{U}_G^d$

$$\begin{tikzcd} \rightarrow & (\mathcal{U}_G^d)^{\mathbb{C}^*} \arrow[d, equal] & \mathcal{A} \arrow[l, "p"'] \arrow[r, "\imath"] \arrow[d, equal] & \mathcal{U}_G^d \\ & \mathcal{U}_L^d & \mathcal{U}_P^d & \end{tikzcd}$$

depending only on L & P

- $\mathcal{U}_L^d \xleftarrow{\cong} \mathcal{U}_{[L,L]}^d$ homeo. no need to make distinction

ρ : generic $\Rightarrow$ $L = T$ torus
$P = B$ Borel

$\mathcal{U}_T^d = S^d \mathbb{A}^2$